ECE 344
Microwave Fundamentals

Spring 2017

Lecture 04:
Rectangular Waveguides

Prepared By

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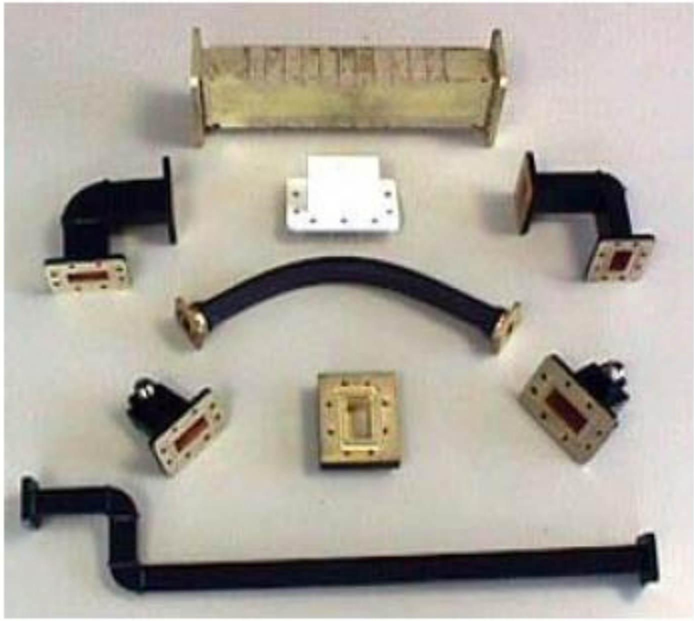
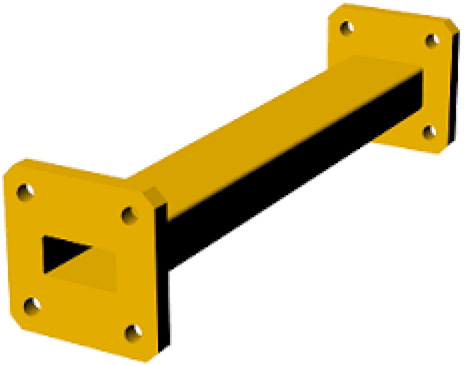
Introduction



Introduction

- Waveguides, like transmission lines, are structures used to guide electromagnetic waves from point to point.
- However, the fundamental characteristics of waveguide and transmission line waves (*modes*) are quite different.
- The differences in these modes result from the basic differences in geometry for a transmission line and a waveguide.



Introduction

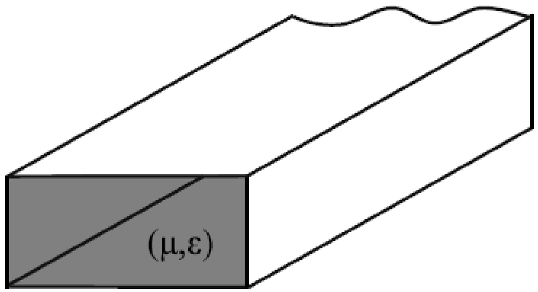
Waveguides can be generally classified as either

- *Metal waveguides*
- *Dielectric waveguides*

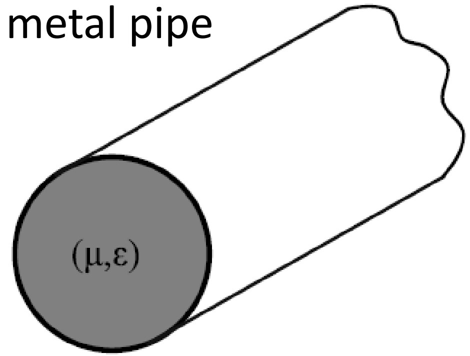
Introduction

Metal Waveguides

enclosed conducting metal pipe



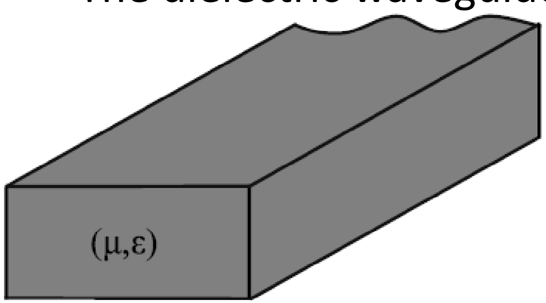
Rectangular Waveguide



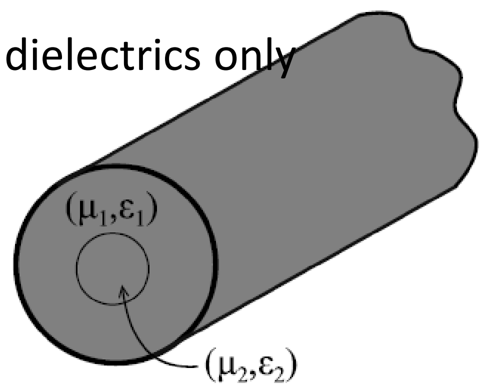
Circular Waveguide

Dielectric Waveguides

The dielectric waveguide consists of dielectrics only



Dielectric Slab Waveguide



Optical Fiber

Introduction

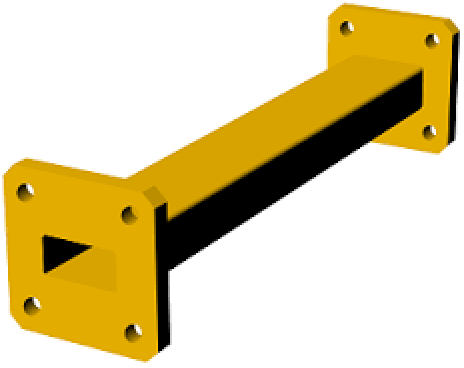
Comparison of Waveguide and Transmission Line Characteristics

T.L	W.G
Two or more conductors separated by some insulating medium (two-wire, coaxial, microstrip, etc.).	Metal waveguides are typically one enclosed conductor filled with an insulating medium (rectangular, circular) Dielectric waveguide consists of multiple dielectrics.
Normal operating mode is the TEM or quasi-TEM mode	Operating modes are TE or TM modes (cannot support a TEM mode).
No cutoff frequency for the TEM mode.	Must operate the waveguide at a frequency above the respective TE or TM mode cutoff frequency for that mode to propagate.
Significant signal attenuation at high frequencies due to conductor and dielectric losses.	Lower signal attenuation at high frequencies than transmission lines.

Introduction

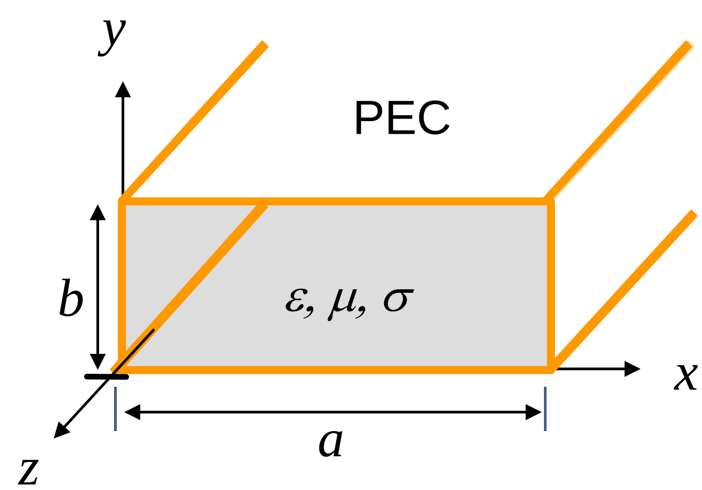
Comparison of Waveguide and Transmission Line Characteristics

T.L	W.G
Small cross-section transmission lines (like coaxial cables) can only transmit low power levels	Metal waveguides can transmit high power levels.
Large cross-section transmission lines (like power transmission lines) can transmit high power levels.	Large cross-section (low frequency) waveguides are impractical due to large size and high cost.



Rectangular Waveguide

- One of the earliest waveguides.
- Still common for high power and low-loss microwave / millimeter-wave applications.



- It is essentially an electromagnetic pipe with a rectangular cross-section.

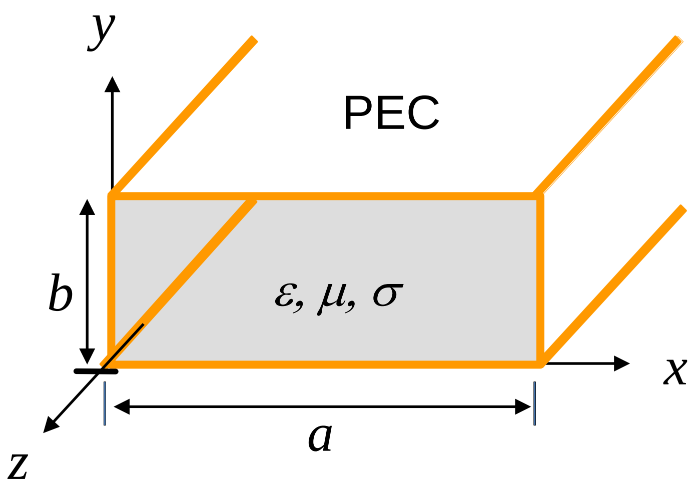
Single conductor $\Rightarrow$ <u>No</u> TEM mode

<u>For convenience</u>

- $a \geq b$.
- The long dimension lies along x.

Rectangular Waveguide

We will now generalize our discussion of transmission lines by considering EM waveguides. These are "pipes" that guide EM waves



Proceeding from the Maxwell curl equations

$$\nabla \times \overline{E} = -j\omega\mu\overline{H} \quad \Rightarrow \quad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu\overline{H}$$

Rectangular Waveguide

$$\hat{x}: \quad \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x$$

$$\hat{y}: \quad -\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) = -j\omega\mu H_y$$

$$\hat{z}: \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z$$

However, the spatial variation in *z* is known so that

$$\frac{\partial\left(e^{-j\beta z}\right)}{\partial z} = -j\beta\left(e^{-j\beta z}\right)$$

Rectangular Waveguide

Consequently, these curl equations simplify to

$$\frac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x \qquad (1)$$

$$-\frac{\partial E_z}{\partial x} - j\beta E_x = -j\omega\mu H_y \qquad (2)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \qquad (3)$$

Similar expansion of Ampere's equation $\nabla \times \overline{H} = j\omega\varepsilon\overline{E}$

$$\frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\varepsilon E_x \qquad (4)$$

$$-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \qquad (5)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \qquad (6)$$

Rectangular Waveguide

Now, (1)-(6) can be manipulated to produce simple algebraic equations for the transverse (x and y) components of $\bar{E}$ and $\bar{H}$.

For example, from (1):

$$H_x = \frac{j}{\omega\mu}\left(\frac{\partial E_z}{\partial y} + j\beta E_y\right)$$

Substituting for E_y from (5) we find

$$H_x = \frac{j}{\omega\mu}\left[\frac{\partial E_z}{\partial y} + j\beta\frac{1}{j\omega\varepsilon}\left(-j\beta H_x - \frac{\partial H_z}{\partial x}\right)\right]$$

$$= \frac{j}{\omega\mu}\frac{\partial E_z}{\partial y} + \frac{\beta^2}{\omega^2\mu\varepsilon}H_x - \frac{j\beta}{\omega^2\mu\varepsilon}\frac{\partial H_z}{\partial x}$$

$$\boxed{H_x = \frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right)} \qquad (7)$$

where $\boxed{k_c^2 \equiv k^2 - \beta^2}$ and $k^2 = \omega^2\mu\varepsilon$

Rectangular Waveguide

Similarly, we can show that

$$\boxed{H_y = -\frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right)} \qquad (8)$$

$$\boxed{E_x = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right)} \qquad (9)$$

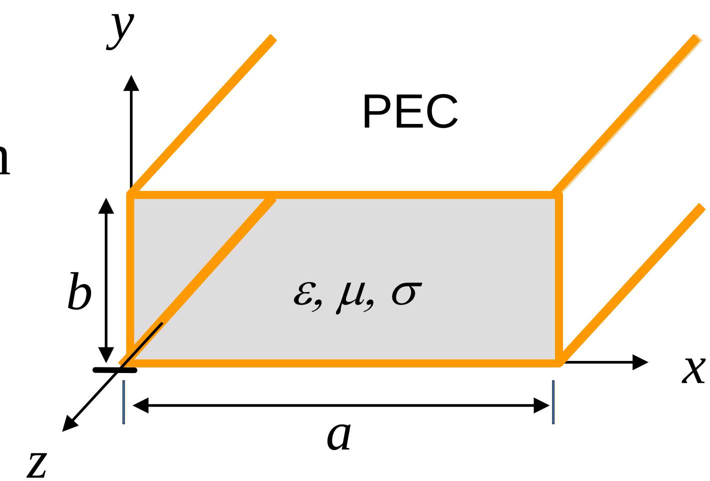
$$\boxed{E_y = \frac{j}{k_c^2}\left(-\beta\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right)} \qquad (10)$$

Most important point: From (7)-(10), we can see that **all transverse** components of $\bar{E}$ and $\bar{H}$ can be determined from **only the axial** components E_z and H_z. It is this fact that allows the mode designations TEM, TE, and TM.

TE$_z$ Modes

A transverse electric (TE) wave has $E_z = 0$ and $H_z \neq 0$.

Hence, in (7)-(10) with $E_z = 0$, the transverse components $\overline{E}$ and $\overline{H}$ are known once we find a solution for only H_z. H_z must satisfy the reduced wave equation of



$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) h_z(x, y) = 0$$

where $H_z(x, y, z) = h_z(x, y) e^{\mp j k_z z}$ $\quad k_c = \left(k^2 - k_z^2\right)^{1/2}$

From Eq. (9), (10):

$$E_x = \frac{-j}{k_c^2}\left(\pm k_z \cancel{\frac{\partial E_z}{\partial x}} + \omega\mu \frac{\partial H_z}{\partial y}\right)$$

$$E_y = \frac{j}{k_c^2}\left(\mp k_z \cancel{\frac{\partial E_z}{\partial y}} + \omega\mu \frac{\partial H_z}{\partial x}\right)$$

Subject to B.C.'s:

$$E_x = 0 \Rightarrow \frac{\partial H_z}{\partial y} = 0 \quad @\, y = 0, b$$

$$E_y = 0 \Rightarrow \frac{\partial H_z}{\partial x} = 0 \quad @\, x = 0, a$$

TE$_z$ Modes (cont.)

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)h_z(x,y)=-k_c^2\,h_z(x,y)$$

Using separation of variables, let $h_z(x,y)=X(x)Y(y)$

$$\Rightarrow \quad Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}=-k_c^2XY$$

Must be a constant

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=-k_c^2$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}=-k_x^2 \quad \text{and} \quad \frac{1}{Y}\frac{d^2Y}{dy^2}=-k_y^2$$

where $k_x^2+k_y^2=k_c^2 \longleftarrow$ Separation equation

TE_z Modes (cont.)

The general solution for h_z can then be written as

$$h_z(x,y) = \overbrace{(A\cos k_x x + B\sin k_x x)}^{X(x)}\overbrace{(C\cos k_y y + D\sin k_y y)}^{Y(y)}$$

Boundary Conditions:
$$\begin{cases} \dfrac{\partial h_z}{\partial y} = 0 \quad @\, y = 0,\, b \quad \text{A} \\ \dfrac{\partial h_z}{\partial x} = 0 \quad @\, x = 0,\, a \quad \text{B} \end{cases}$$

A $\Rightarrow$ $D = 0$ and $k_y = \dfrac{n\pi}{b}$ $n = 0,1,2,\ldots$

B $\Rightarrow$ $B = 0$ and $k_x = \dfrac{m\pi}{a}$ $m = 0,1,2,\ldots$

$\Rightarrow$ $h_z(x,y) = A_{mn}\cos\left(\dfrac{m\pi x}{a}\right)\cos\left(\dfrac{n\pi y}{b}\right)$ and $k_c^2 = \left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2$

TE$_z$ Modes (cont.)

Therefore,

$$H_z = A_{mn} \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

$$k_z = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

From the previous field-representation equations, we can obtain the following:

$$E_x = \frac{j\omega\mu n\pi}{k_c^2 b} A_{mn} \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

$$E_y = -\frac{j\omega\mu m\pi}{k_c^2 a} A_{mn} \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

$$H_x = \pm\frac{jk_z m\pi}{k_c^2 a} A_{mn} \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

$$H_y = \pm\frac{jk_z n\pi}{k_c^2 b} A_{mn} \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

Note:

$$m = 0,1,2,\ldots$$
$$n = 0,1,2,\ldots$$

But $m = n = 0$ is not allowed!

(non-physical solution)

$$\underline{H} = \underline{\hat{z}}\, A_{00} e^{\mp jkz};\ \nabla \cdot \underline{H} \neq 0$$

TE_z Modes (cont.)

Lossless case $\left(\varepsilon_c = \varepsilon = \varepsilon'\right)$

$$k_z^{mn} = \sqrt{k^2 - \left(k_c^{mn}\right)^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$\Rightarrow \mathrm{TE}_{mn}$ mode is at cutoff when $k = k_c^{mn}$

$$f_c^{mn} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

The mode with the lowest cutoff frequency is called the dominant mode;

For $(a > b)$, Lowest cutoff frequency is for TE_{10} mode

We will revisit this mode later.

$$f_c^{10} = \frac{1}{2a\sqrt{\mu\varepsilon}}$$

Dominant TE mode (lowest f_c)

TE_z Modes (cont.)

At the cutoff frequency of the TE_{10} mode (lossless waveguide):

$$f_c^{10} = \frac{1}{2a\sqrt{\mu\varepsilon}}$$

Minimum dimension of a for TE_{10} mode to propagate

$$\Rightarrow \lambda_d = \frac{c_d}{f} = \frac{c_d}{f_c^{10}} = \frac{c_d}{\dfrac{1}{2a\sqrt{\mu\varepsilon}}} = 2a$$

so $$\boxed{a\big|_{f=f_c} = \lambda_d / 2}$$

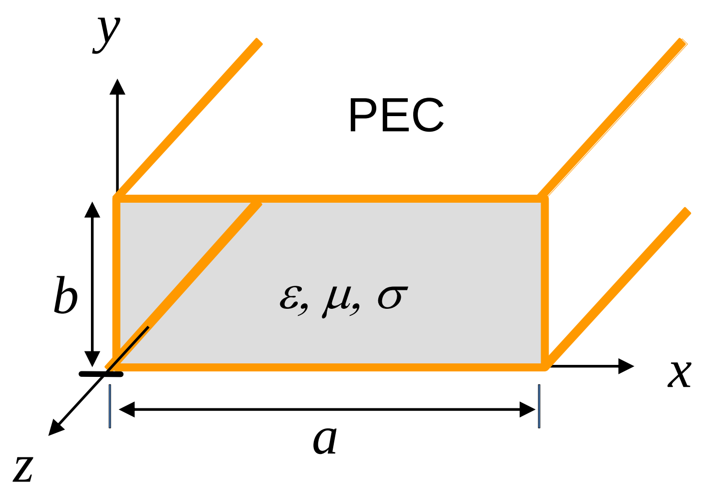
For a given frequency (with $f > f_c$) , the dimension a must be at least $\lambda_d / 2$ in order for the TE_{10} mode to propagate.

Example: Air-filled waveguide, $f = 10$ GHz. We have that $a > 3.0$ cm/2 = 1.5 cm.

TM$_z$ Modes

Recall:

$$E_z(x,y,z) = e_z(x,y)e^{\mp jk_z z}$$



where

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)e_z(x,y) = -k_c^2 e_z(x,y)$$

$$k_c = \left(k^2 - k_z^2\right)^{1/2}$$

Subject to B.C.'s: $E_y = 0 \quad @ \quad x = 0,\ a$

$$E_x = 0 \quad @ \quad y = 0,\ b$$

Thus, following same procedure as before, we have the following result:

TM$_z$ Modes (cont.)

$$e_z(x,y) = \overbrace{(A\cos k_x x + B\sin k_x x)}^{X(x)}\overbrace{(C\cos k_y y + D\sin k_y y)}^{Y(y)}$$

Boundary Conditions: $\dfrac{\partial E_z}{\partial y} = 0 \quad @\, y = 0,\, b$ A

$\dfrac{\partial E_z}{\partial x} = 0 \quad @\, x = 0,\, a$ B

A $\Rightarrow \quad C = 0$ and $k_y = \dfrac{n\pi}{b} \quad n = 0,1,2,\ldots$

B $\Rightarrow \quad A = 0$ and $k_x = \dfrac{m\pi}{a} \quad m = 0,1,2,\ldots$

$\Rightarrow$ $e_z = B_{mn}\sin\left(\dfrac{m\pi}{a}x\right)\sin\left(\dfrac{n\pi}{b}y\right)$ and $k_c^2 = \left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2$

TM$_z$ Modes (cont.)

Therefore

$$E_z = B_{mn} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

$$k_z = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

From the previous field-representation equations, we can obtain the following:

$$m = 1,2,3,\ldots$$
$$n = 1,2,3,\ldots$$

$$H_x = \frac{j\omega\varepsilon_c n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

$$H_y = -\frac{j\omega\varepsilon_c m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

$$E_x = \mp\frac{jk_z m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

$$E_y = \pm\frac{jk_z n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

Note: If either m or n is zero, the field becomes a trivial one in the TM$_z$ case.

TM_z Modes (cont.)

Lossless case $\left(\varepsilon_c = \varepsilon = \varepsilon'\right)$

$$k_z^{mn} = \sqrt{k^2 - \left(k_c^{mn}\right)^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

(same as for TE modes)

$$f_c^{mn} = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

The lowest cutoff frequency is obtained for the TM_{11} mode

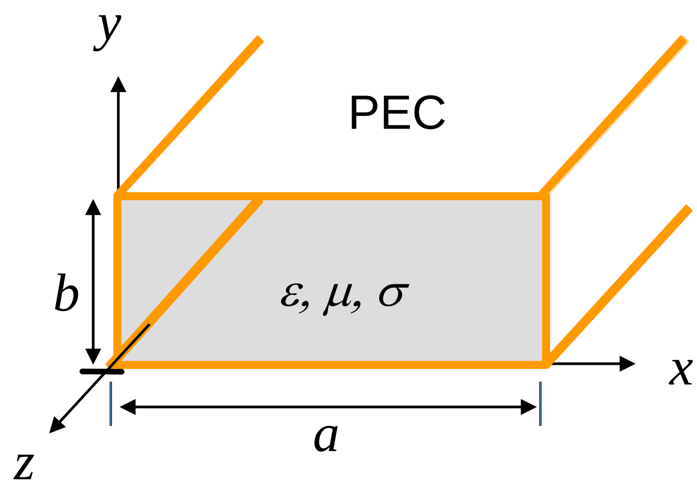
$$f_c^{11} = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

Dominant TM mode
(lowest f_c)

- Note that the TE_{10} and TE_{01} modes are *degenerate modes* (modes with the same cutoff frequency) for a square waveguide.
- The rectangular waveguide allows one to operate at a frequency above the cutoff of the dominant TE_{10} mode but below that of the next highest mode to achieve single mode operation.
- A waveguide operating at a frequency where more than one mode propagates is said to be *overmoded*.

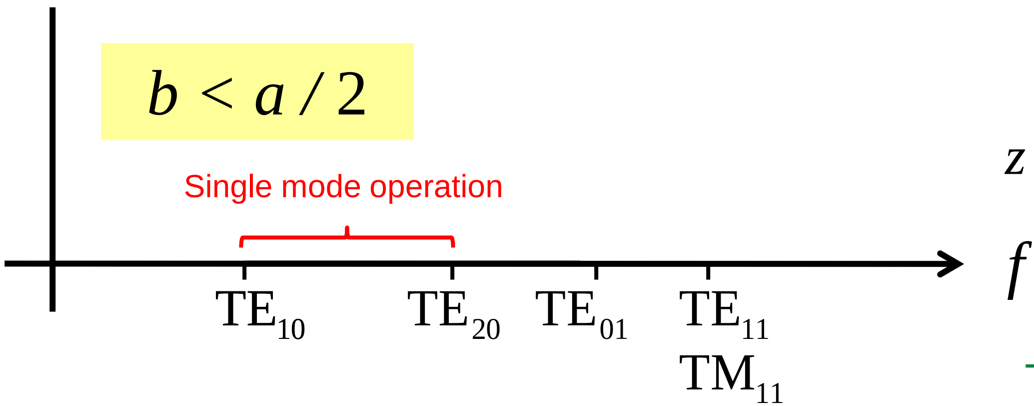
Mode Chart

Lossless case $(\varepsilon_c = \varepsilon = \varepsilon')$



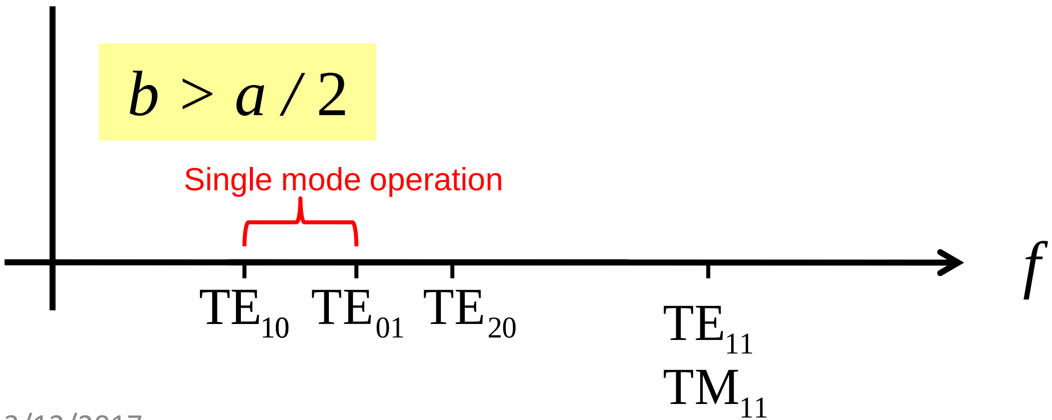
$a > b$

Two cases are considered:



The maximum band for single mode operation is f_c^{10} [Hz] (67% bandwidth).

$$\left(b \le a/2\right)$$

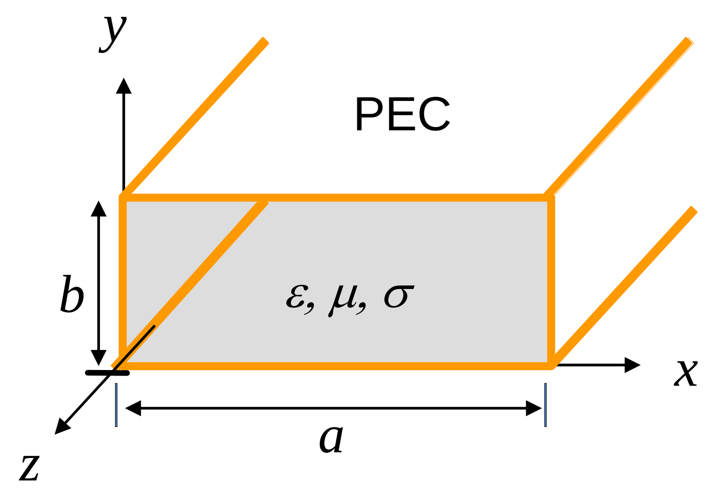


$$f_c^{mn} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

Dominant Mode: TE_{10} Mode

For this mode we have

$$m = 1,\ n = 0,\ k_c = \frac{\pi}{a}$$



$$k_z = k_z^{10} = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}$$

Hence we have

$$H_z = A_{10} \cos\left(\frac{\pi}{a} x\right) e^{\mp jk_z z}$$

$$H_x = \pm j \frac{k_z a}{\pi} A_{10} \sin\left(\frac{\pi}{a} x\right) e^{\mp jk_z z}$$

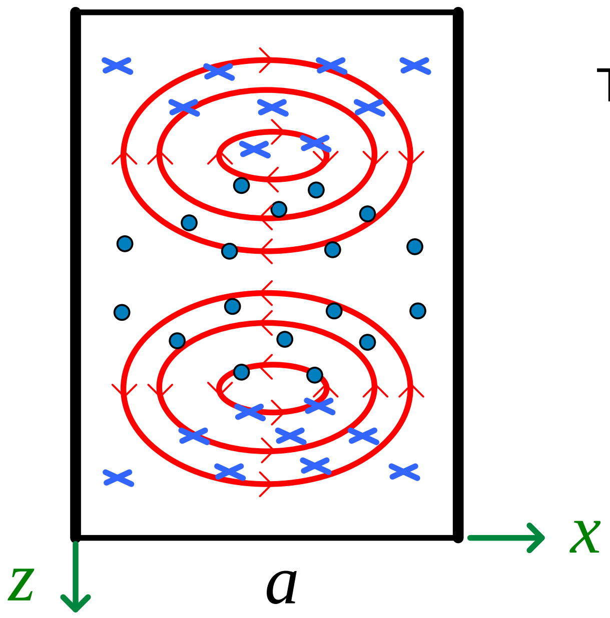
$$E_y = \underbrace{-\frac{j\omega\mu a}{\pi} A_{10}}_{E_{10}} \sin\left(\frac{\pi}{a} x\right) e^{\mp jk_z z}$$

$$E_y = E_{10} \sin\left(\frac{\pi}{a} x\right) e^{\mp jk_z z}$$

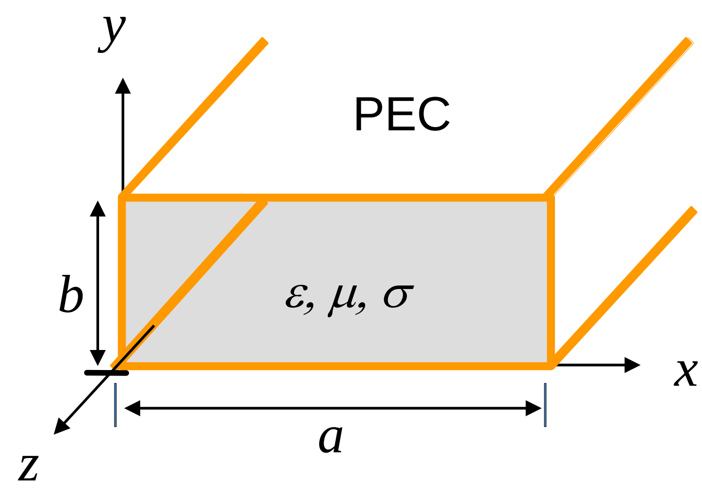
$$A_{10} \equiv \frac{-\pi}{j\omega\mu a} E_{10}$$

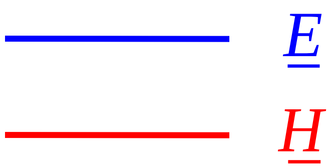
$$E_x = E_z = H_y = 0$$

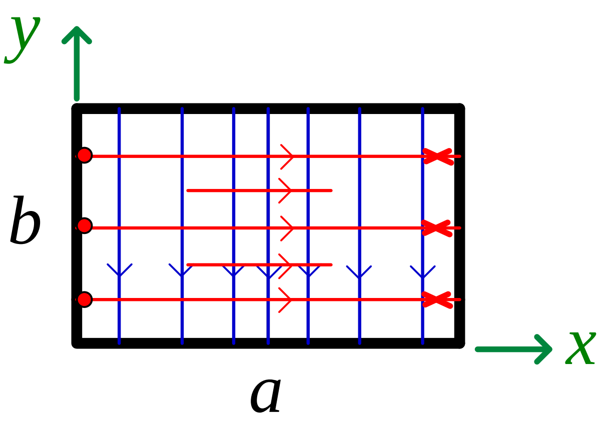
Field Plots for TE_{10} Mode



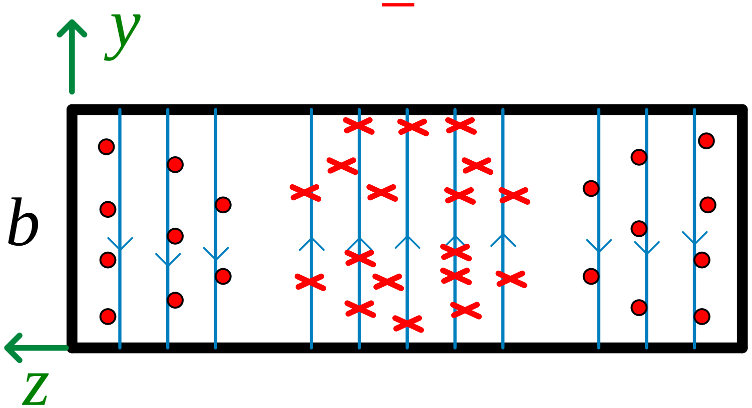
Top view







End view

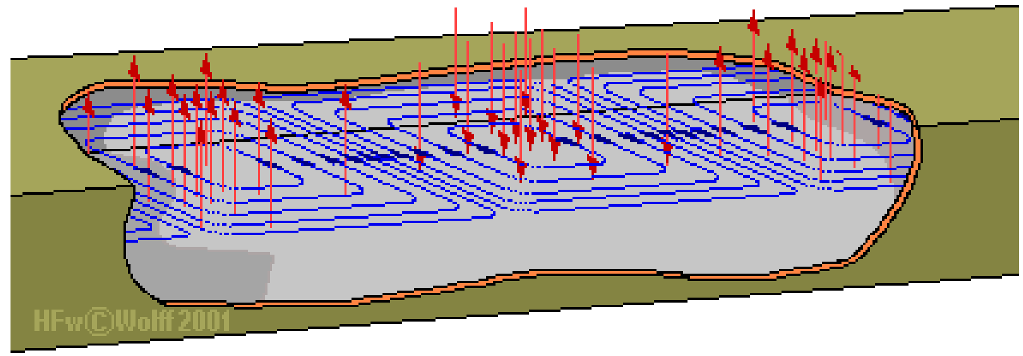


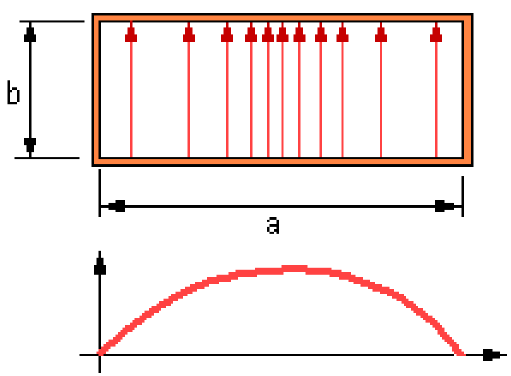
Side view

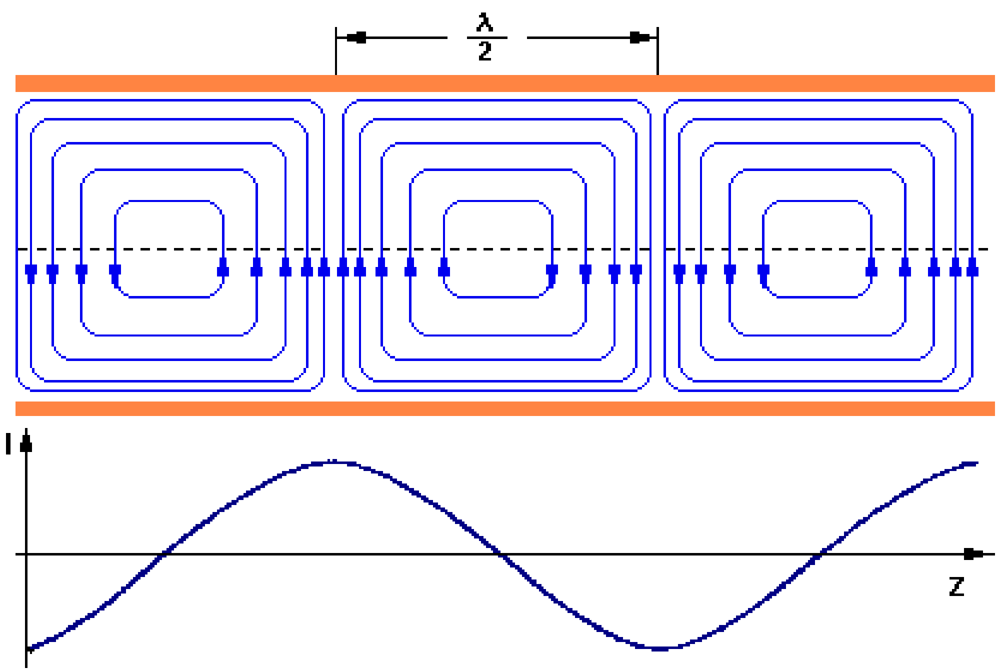
TE10 Mode

Mode with lowest cutoff frequency is dominant mode

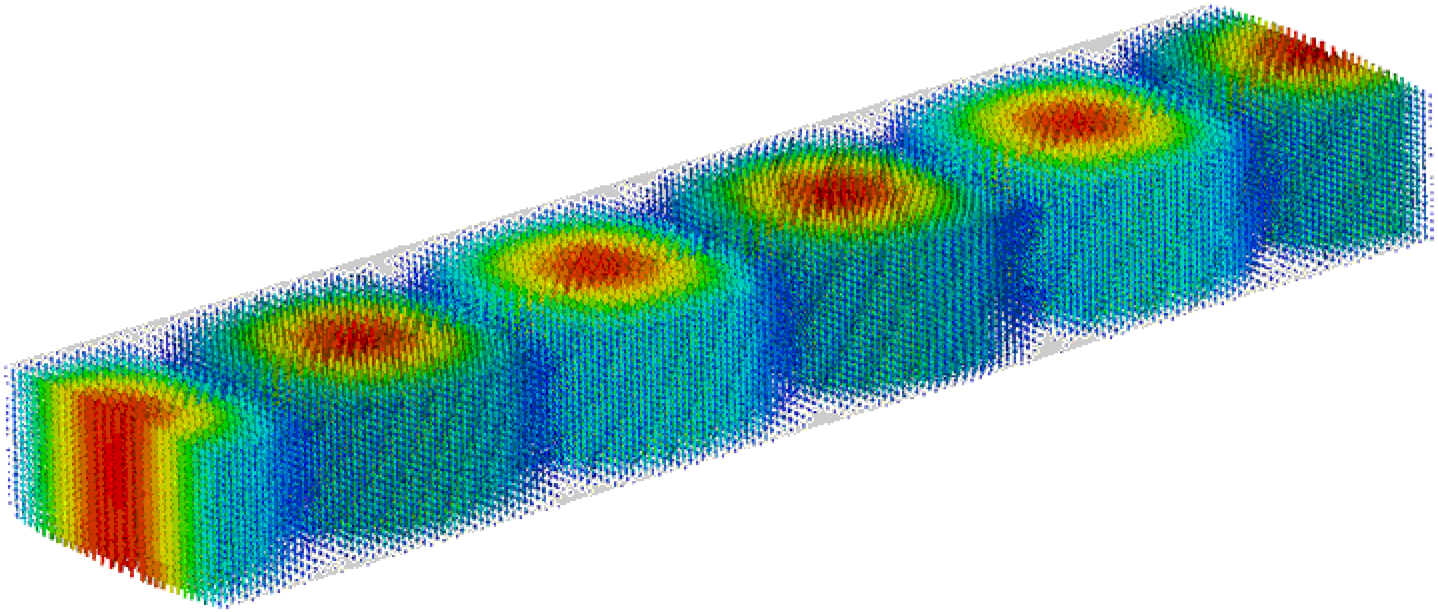
Single mode propagation is highly desirable to reduce dispersion



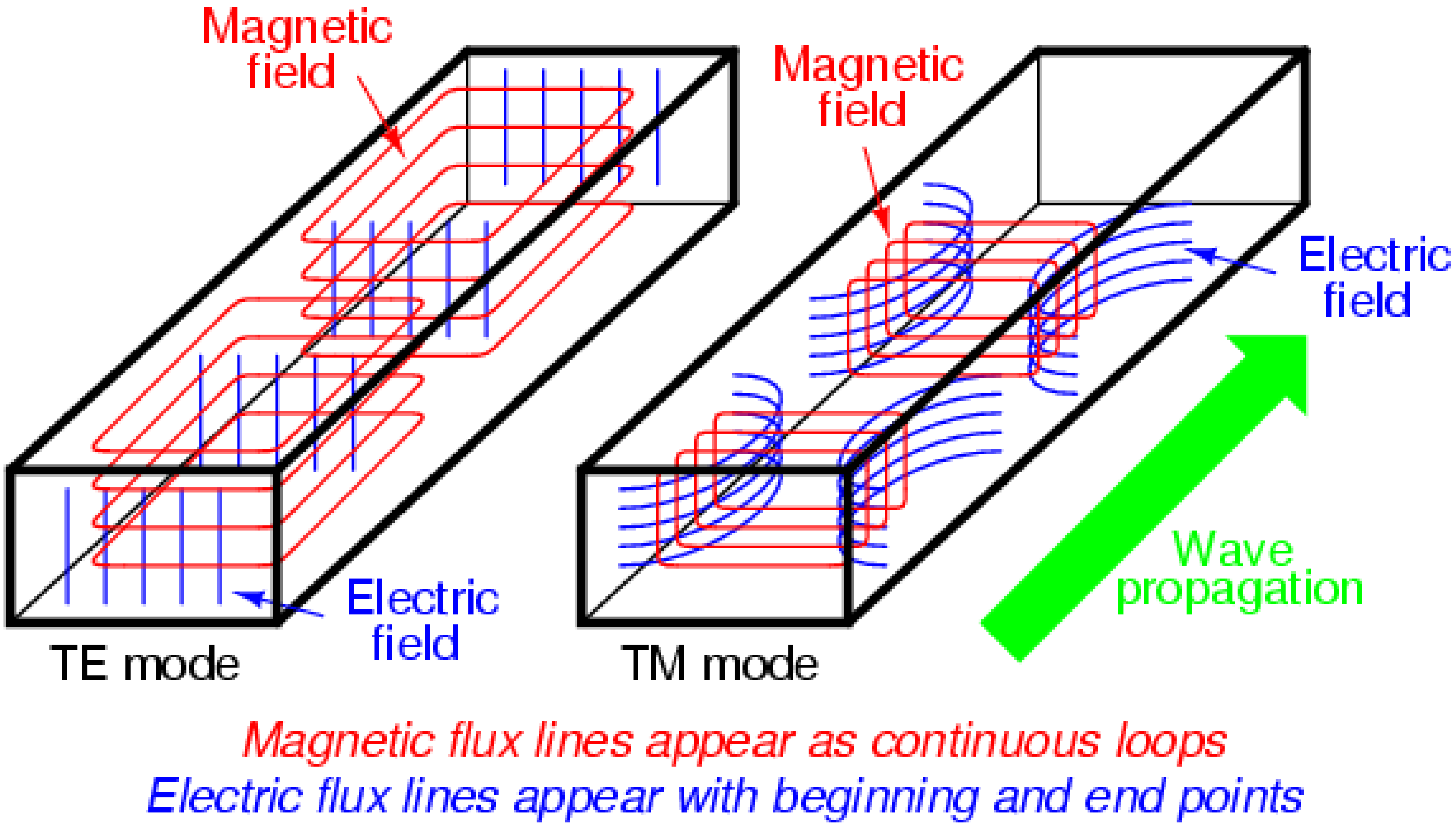




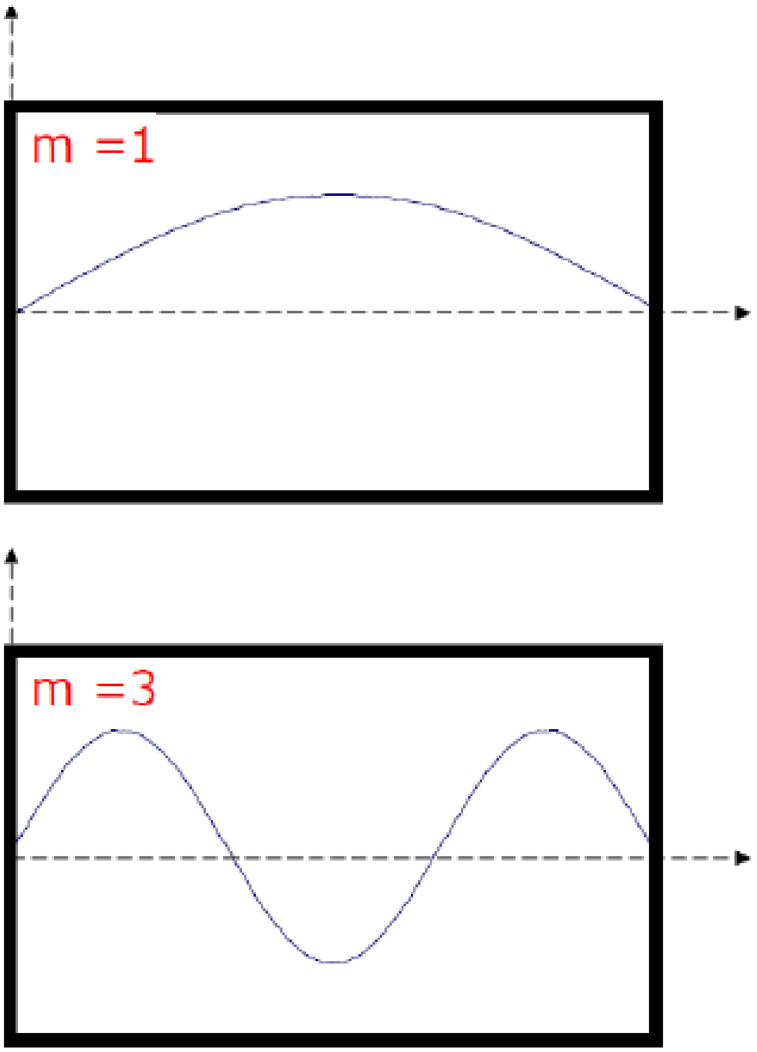
TE10 Mode



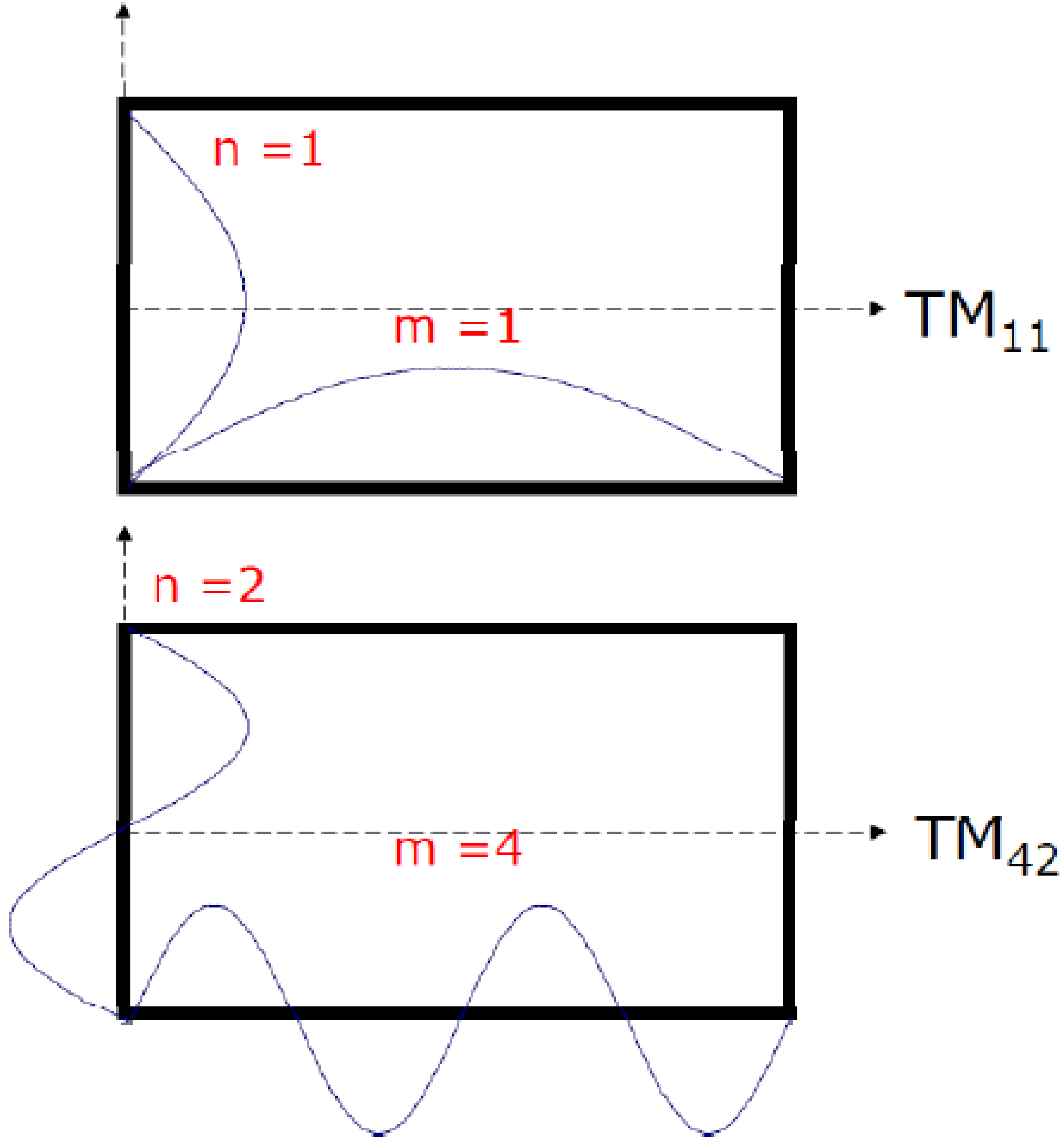
E-field phase animation inside a rectangular waveguide at 10 GHz.

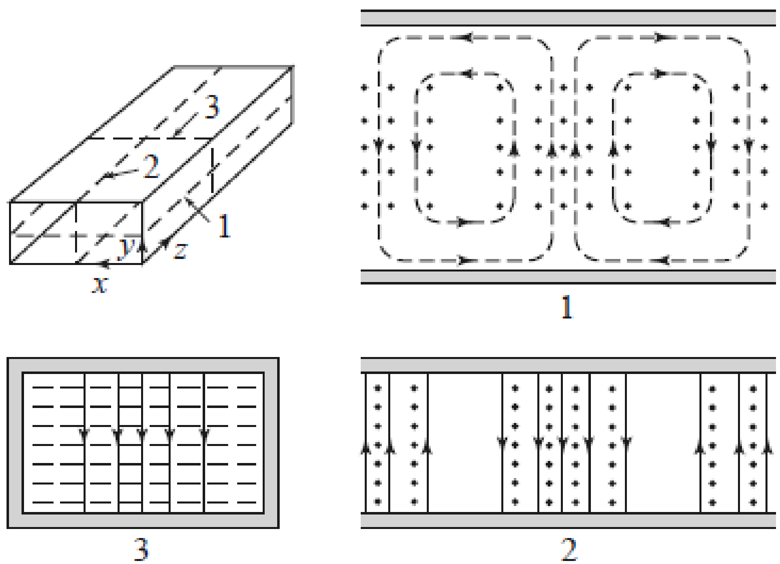
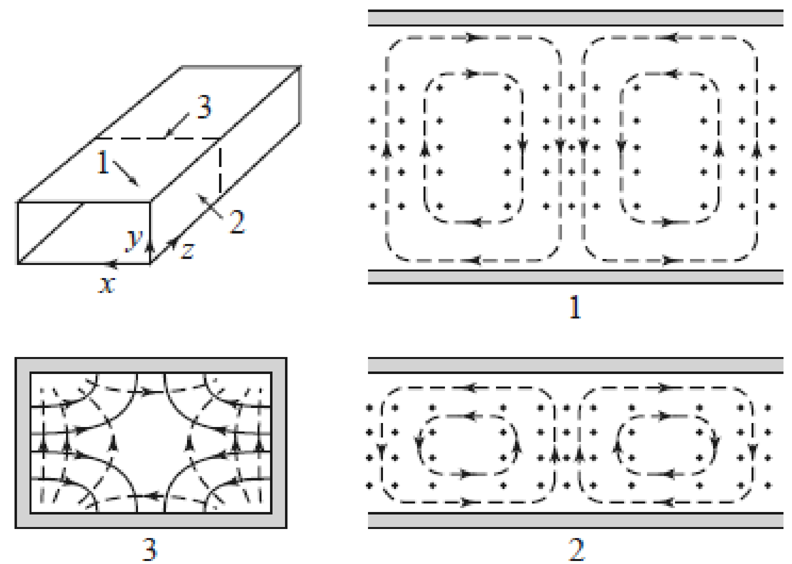

Magnetic field
Electric field
TE mode
Magnetic field
Electric field
TM mode
Wave propagation
Magnetic flux lines appear as continuous loops
Electric flux lines appear with beginning and end points

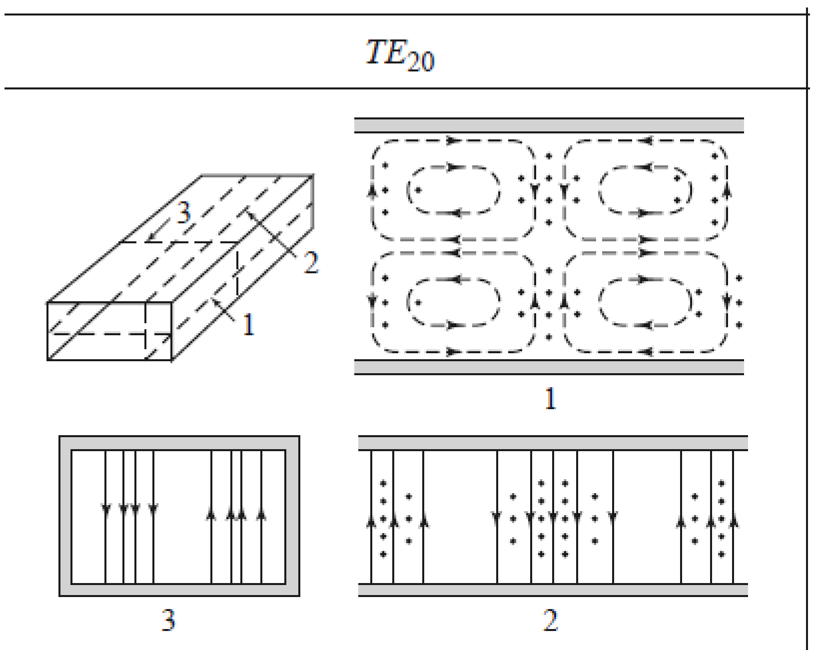
Field Components

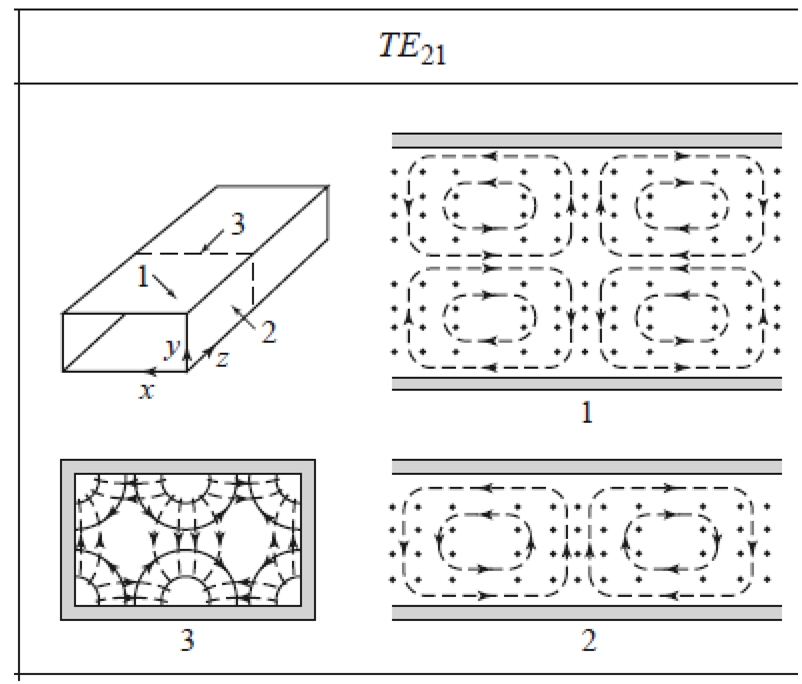


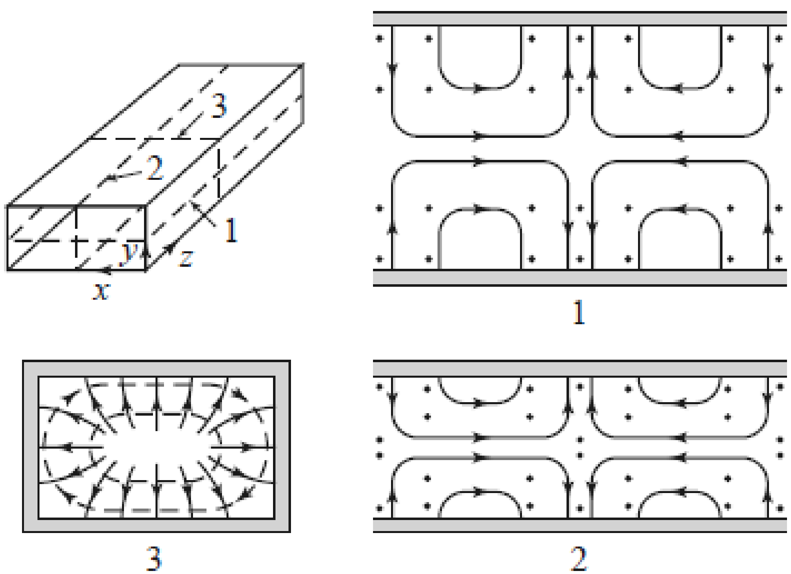
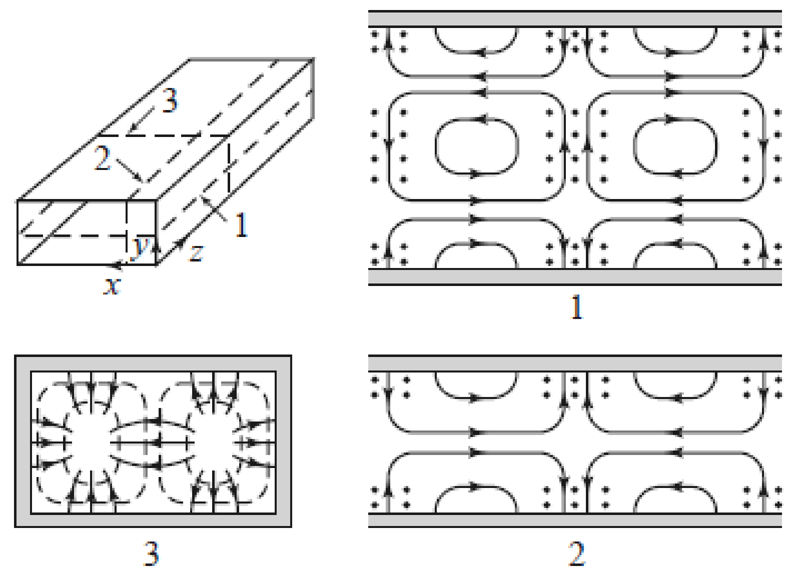
Field Components



TE_{10}	TE_{11}
	


TE_{20}
3
2
1
1
3
2


TE_{21}
3
1
2
y
z
x
1
3
2

TM_{11}	TM_{21}
	

The velocity of propagation for a TEM wave (plane wave or transmission line wave) is referred to as the *phase velocity* (the velocity at which a point of constant phase moves).

$$u_p' = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \text{(TEM phase velocity)}$$

$$\beta_{mn} = k\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}$$

$$f_{c_{mn}} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$$

$$\lambda_{c_{mn}} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}}$$

Guided wavelength

$$\lambda_{mn} = \frac{2\pi}{\beta_{mn}} = \frac{2\pi}{k\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}} = \frac{\lambda'}{\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}}$$

$$\lambda' = \frac{2\pi}{k} \qquad \text{(TEM wavelength)}$$

$$k^2 = \beta^2 + k_c^2 \longrightarrow \left(\frac{2\pi}{\lambda}\right)^2 = \left(\frac{2\pi}{\lambda_g}\right)^2 + \left(\frac{2\pi}{\lambda_c}\right)^2 \longrightarrow \left(\frac{1}{\lambda}\right)^2 = \left(\frac{1}{\lambda_g}\right)^2 + \left(\frac{1}{\lambda_c}\right)^2$$

$$\left(\frac{\lambda}{\lambda_g}\right)^2 = 1 - \left(\frac{\lambda}{\lambda_c}\right)^2 \qquad \lambda_g = \frac{\lambda}{\sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^2}}$$

$$\eta_{TE_{mn}} = \frac{\tilde{E}_x^{TE_{mn}}}{\tilde{H}_y^{TE_{mn}}} = -\frac{\tilde{E}_y^{TE_{mn}}}{\tilde{H}_x^{TE_{mn}}} = \frac{\omega\mu}{\beta_{mn}} = \frac{\omega\mu}{k\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}}$$

$$\eta_{TM_{mn}} = \frac{\tilde{E}_x^{TM_{mn}}}{\tilde{H}_y^{TM_{mn}}} = -\frac{\tilde{E}_y^{TM_{mn}}}{\tilde{H}_x^{TM_{mn}}} = \frac{\beta_{mn}}{\omega\varepsilon} = \frac{k}{\omega\varepsilon}\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}$$

$$\frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}} = \eta' \qquad \frac{k}{\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}} = \eta' \qquad \left(\begin{array}{c}\text{TEM instrinsic} \\ \text{wave impedance}\end{array}\right)$$

$$\eta_{TE_{mn}} = \frac{\eta'}{\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}} \qquad \eta_{TM_{mn}} = \eta'\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}$$

Note that the product of the TE and TM wave impedances is equal to the square of the TEM wave impedance = intrinsic impedance of the material filling the waveguide.

$$\eta_{TM_{mn}}\eta_{TE_{mn}} = \eta'^2$$

Waveguide Group Velocity and Phase Velocity

The phase velocity of a TEM wave traveling in a lossless medium characterized by (μ, ε) is given by

$$u_p' = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \text{(TEM phase velocity)}$$

For the general TE_{mn} of TM_{mn} waves,

$$u_{p_{mn}} = \frac{\omega}{\beta_{mn}}$$

$$u_{p_{mn}} = \frac{\omega}{\beta_{mn}} = \frac{\omega}{k\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}} = \frac{u_p'}{\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}}$$

Waveguide Group Velocity and Phase Velocity

The group velocity Is the velocity at which the energy travels.

$$u_g = \frac{1}{d\beta / d\omega},$$

$$\beta_{mn} = k\sqrt{1-\left(\frac{\omega_{c_{mn}}}{\omega}\right)^2} = \frac{k}{\omega}\sqrt{\omega^2 - \omega_{c_{mn}}{}^2} = \frac{1}{u_p^{'}}\sqrt{\omega^2 - \omega_{c_{mn}}{}^2}$$

$$u_{g_{mn}} = u_p^{'}\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2} = \frac{\lambda^{'}}{\lambda_{g_{mn}}}u_p^{'}$$

$$u_{p_{mn}} u_{g_{mn}} = u_p^{'2}$$

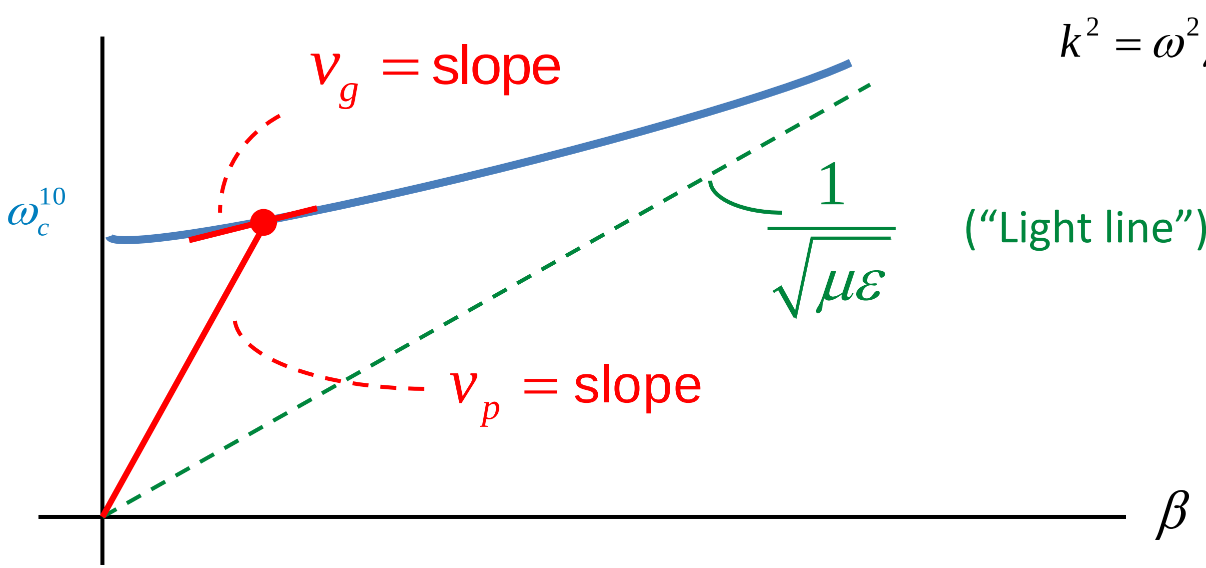
Dispersion Diagram for TE_{10} Mode

Lossless case $\left(\varepsilon_c = \varepsilon = \varepsilon'\right)$

$$k_z = \beta = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}$$

$$k^2 = \omega^2 \mu \varepsilon$$

$f > f_c$

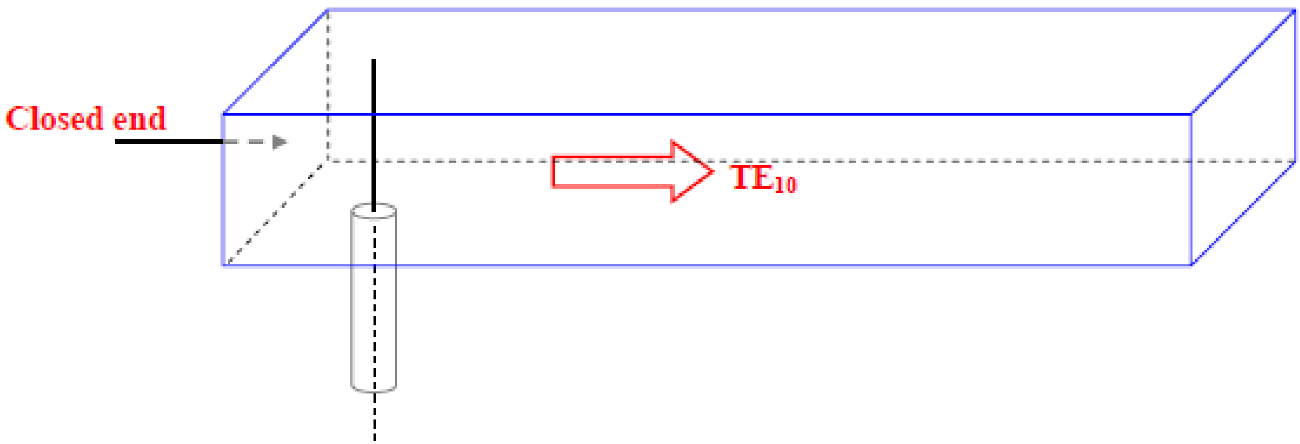


Phase velocity: $$v_p = \frac{\omega}{\beta}$$

Group velocity: $$v_g = \frac{d\omega}{d\beta}$$

Velocities are slopes on the dispersion plot.

- The simple arrangement below can be used to excite the TE_{10} in a rectangular waveguide.



The inner conductor of the coaxial cable behaves like an antenna and it creates a maximum electric field in the middle of the cross-section.

Solved Problems

Example – Design an air-filled rectangular wave guide for the following operation conditions:

a. 10 GHz in the middle of the frequency band (single mode operation)

b. b=a/2

The fundamental mode is the TE_{10} with cut-off frequency

$$f_c(TE_{10}) = \frac{1}{2a\sqrt{\mu_o \varepsilon_o}} = \frac{c}{2a} \approx \frac{3\times 10^8\, m/\sec}{2a} Hz$$

For b=a/2, TE_{01} and TE_{20} have the same cut-off frequency

$$f_c(TE_{01}) = \frac{1}{2b\sqrt{\mu_o \varepsilon_o}} = \frac{c}{2b} = \frac{c}{2}\frac{2}{a} = \frac{c}{a} \approx \frac{3\times 10^8\, m/\sec}{a} Hz$$

$$f_c(TE_{20}) = \frac{1}{a\sqrt{\mu_o \varepsilon_o}} = \frac{c}{a} \approx \frac{3\times 10^8\, m/\sec}{a} Hz$$

Solved Problems

The operation frequency can be expressed in terms of the cut-off frequencies

$$f = f_c(TE_{10}) + \frac{f_c(TE_{10}) - f_c(TE_{01})}{2}$$

$$= \frac{f_c(TE_{10}) + f_c(TE_{01})}{2} = 10.0GHz$$

$$\Rightarrow 10.0 \times 10^9 = \frac{1}{2}\left[\frac{3\times10^8}{2a} + \frac{3\times10^8}{a}\right]$$

$$\Rightarrow a = 2.25cm \quad b = \frac{a}{2} = 1.125cm$$

Solved Problems

We consider an air filled guide, so $\varepsilon_r=1$. The internal size of the guide is 0.9 x 0.4 inches (waveguides come in standard sizes). The cut-off frequency of the dominant mode:

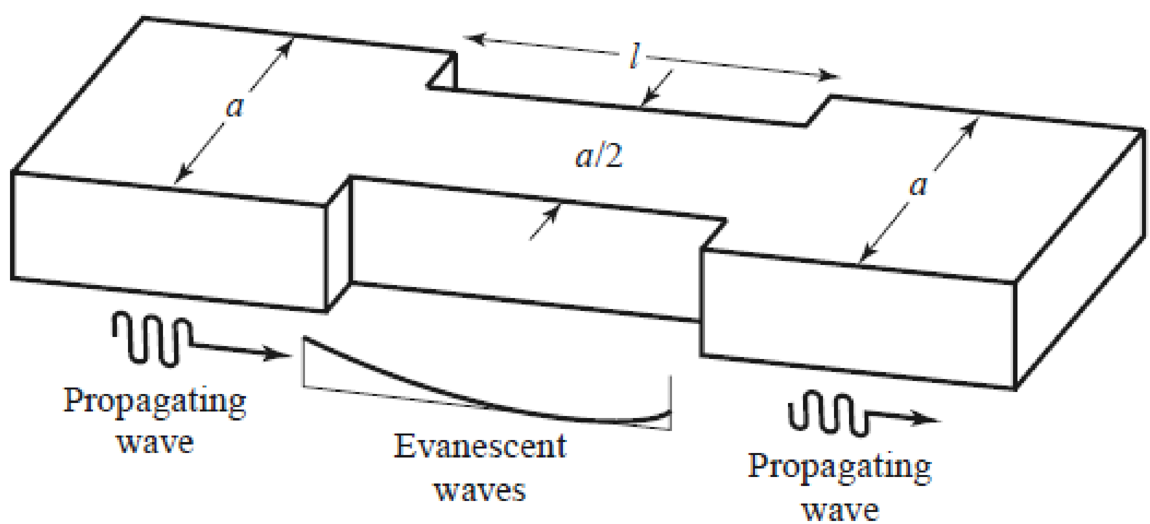
$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \qquad \mathrm{a} = 0.9'' = 22.86\mathrm{mm};\ \mathrm{b} = 0.4'' = 10.16\mathrm{mm}$$

$$\left(k_c\right)_{TE_{10}} = \frac{\pi}{a} = 137.43$$

$$\left(f_c\right)_{TE_{10}} = \frac{k_c}{2\pi\sqrt{\mu_0\varepsilon_0}} = \frac{137.43\times3\times10^8}{2\pi} = 6.56GHz$$

Solved Problems

An attenuator can be made using a section of waveguide operating below cutoff, as shown in the accompanying figure. If a = 2.286 cm and the operating frequency is 12 GHz, determine the required length of the below-cutoff section of waveguide to achieve an attenuation of 100 dB between the input and output guides. Ignore the effect of reflections at the step discontinuities.



Solved Problems

Solution:
In the section of WG of width a/2, the TE_{10} mode is below cutoff (evanescent),with an attenuation constant α

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c/f} = 251.3\,\mathrm{m}^{-1}$$

$$\alpha = \sqrt{\left(\frac{\pi}{a/2}\right)^2 - k^2} = \sqrt{\left(\frac{2\pi}{0.02286}\right)^2 - (251.3)^2} = 111.3\,\text{neper/m}$$

To obtain 100 dB attenuation (ignoring reflection)

$$-100\mathrm{dB} = 20\log(\mathrm{e}^{-\alpha l})$$

$$10^{-5} = \mathrm{e}^{-\alpha l}$$

$$l = \frac{11.5}{111.3} = 0.103\,\mathrm{m}$$

Solved Problems

A rectangular waveguide measures 3×4.5cm internally and has a 10 GHz signal propagated in it. Calculate the cut off frequency (λ_c) and the guide wavelength (λ_g).

(4)

Ans: $\lambda_c = \dfrac{2}{\sqrt{\left(\dfrac{m}{a}\right)^2 + \left(\dfrac{n}{b}\right)^2}}$

TE_{10} mode m = 1, n = 0

$$\lambda_c = \frac{2a}{m} = 2\times0.045 = 0.090m$$

$$\lambda_g = \frac{\lambda}{\sqrt{1-\left(\dfrac{\lambda}{\lambda_c}\right)^2}} \quad \text{where} \quad \lambda = \frac{3\times10^8}{10\times10^9} = 0.03m$$

$$\lambda_g = 0.0318m.$$

In a rectangular waveguide for which a = 1.5 cm, b = 0.8 cm, $\sigma = 0$, $\mu = \mu_0$ and $\epsilon = 4\epsilon_0$ and

$$H_x = 2\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{3\pi y}{b}\right)\mathrm{Sin}\left(\pi\times10^{11}t-\beta z\right)A/m$$

Find

(i) the mode of operation
(ii) the cut off frequency
(iii) phase constant
(iv) the propagation constant
(v) the intrinsic wave impedance

Ans:

(i) TM_{13} or TE_{13}

(ii) $f_{c_{mn}} = \frac{\mu'}{2}\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$, $\mu' = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{2}$

Or $f_{c_{13}} = 28.57 GH_Z$

(iii) $\beta = W\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$

$$= \frac{W\sqrt{\epsilon r}}{C}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$\beta = 1718.81$ rad/m

Where $W = 2\pi f = \pi\times10^{11}$ or $f = \frac{100}{2} = 50GH_Z$

(iv) $\gamma = j\beta = j1718.81/m$

(v) $\eta_{TM_{13}} = \eta\sqrt{1-\left(\frac{f_c}{f}\right)^2} = \frac{377}{\sqrt{\epsilon_r}}\sqrt{1-\left(\frac{28.57}{50}\right)^2} = 154.7\Omega$

Solved Problems

Find the following:

(i) the possible transmission modes in a hollow rectangular waveguide of inner dimension 3.44 × 7.22 cm at an operating frequency of 3000 MHz.

(ii) the corresponding values of phase velocity, group velocity and phase constant.

(7)

Ans: Free space wave length, $\lambda_0 = \dfrac{c}{f} = \dfrac{3\times10^8}{3000\times10^6}$ metre

$= 10$ cm

Also, we know that the cut off wave length

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

Possible modes:

(i) For TE_{00} mode i.e. m = 0, n = 0, $\lambda_c = \infty$, (i.e. $\lambda_c > \lambda_0$) and hence there will be no propagation.

Solved Problems

(ii) For TE_{10} mode i.e. m = 1, n = 0 and $\lambda_c = 2a = 2\times 7.22 = 14.44$ cms. Hence this mode will propagate because $\lambda_c > \lambda_0$.

(iii) For TE_{01} mode i.e. m =0, n = 1, and $\lambda_c = 2b = 2\times 3.44 = 6.88$ cm. Hence this mode will not propagate because $\lambda_c < \lambda_0$.

Obviously the higher TE mode will not propagate for $\lambda_c < \lambda_0$ for other values of m & n. Also for TM_{mn} mode the lowest value of m & n is unity hence no TM mode is possible at the frequency.

Since the guide wave length $\lambda_g = \dfrac{\lambda_0}{\sqrt{1-\left(\dfrac{\lambda_c}{\lambda_0}\right)^2}} = \dfrac{0.1}{\sqrt{1-\left(\dfrac{0.1}{0.144}\right)^2}} = 0.141$ metres.

We get phase velocity $v_p = \left(\dfrac{\lambda_g}{\lambda_0}\right)c = \dfrac{0.141}{0.100}\times 3\times 10^8 = 4.23\times 10^8$ m/sec.

And group velocity, $v_g = \left(\dfrac{\lambda_0}{\lambda_g}\right)c = \dfrac{0.100}{0.141}\times 3\times 10^8 = 2.28\times 10^8$ m/sec.

The phase constant, $\beta = \dfrac{2\pi}{\lambda_g} = \dfrac{2\times 3.14}{0.141} = 44.7$.